

# Digital

18/4/2016 الأسبوع

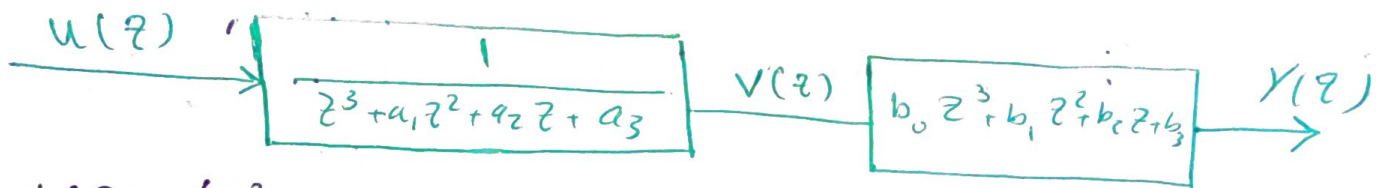
د. عرفة

محاضرة [8]

Quiz

$$T.F. = \frac{z^3 + 2z + 3}{z^3 + 4z^2 + 2z - 1}$$

3rd order System



$$U(z) = (z^3 + a_1z^2 + a_2z + a_3)V(z)$$

$$\xrightarrow{z^{-1}T} u(k) = v(k+3) + a_1v(k+2) + a_2v(k+1) + a_3v(k)$$

$$x_1(k) = v(k)$$

$$x_2(k) = v(k+1) = x_1(k+1)$$

$$x_3(k) = v(k+2) = x_2(k+1)$$

$$x_3(k+1) = v(k+3)$$

$$x_3(k+1) = v(k+3)$$

$$v(k+3) = -a_3v(k) - a_2v(k+1) - a_1v(k+2) + u(k)$$

$$x_3(k+1) = -a_3x_1(k) - a_2x_2(k) - a_1x_3(k) + u(k)$$

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = ( \quad ) x(k) + D u(k)$$

$$\frac{Y(z)}{V(z)} = b_0z^3 + b_1z^2 + b_2z + b_3$$

$$Y(z) = (b_0z^3 + b_1z^2 + b_2z + b_3)V(z)$$

$$\Downarrow z^{-1}T$$

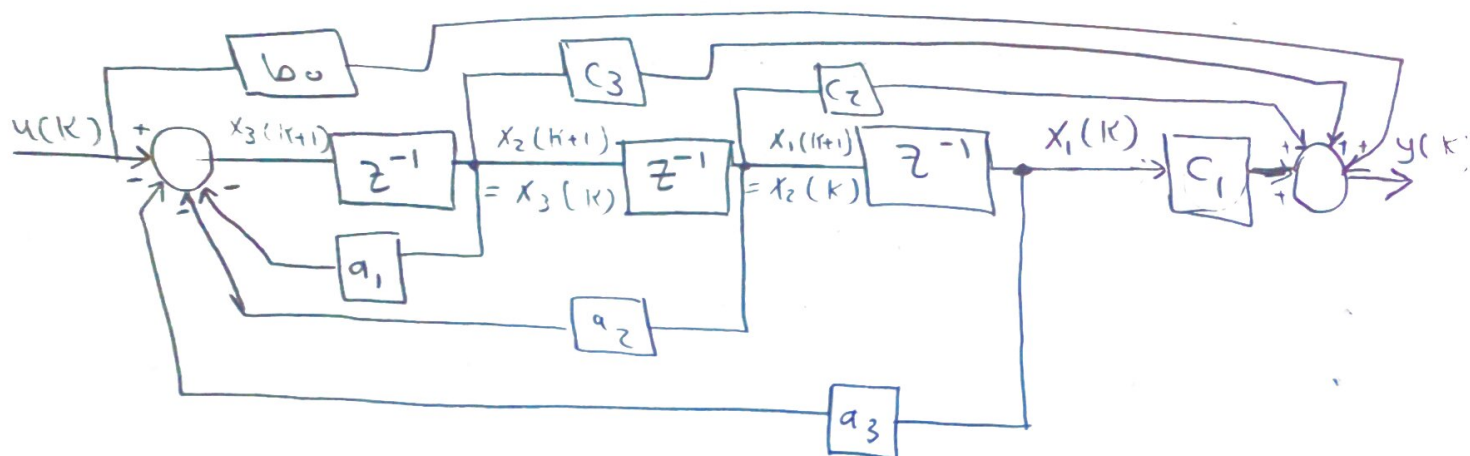
$$y(k) = \underbrace{b_0 v(k+3)}_{x_3(k+1)} + \underbrace{b_1 v(k+2)}_{x_3(k)} + \underbrace{b_2 v(k+1)}_{x_2(k)} + \underbrace{b_3 v(k)}_{x_1(k)}$$

$$y(k) = b_0 (-a_3 x_1(k) - a_2 x_2(k) - a_1 x_3(k) + u(k)) \\ + b_1 x_3(k) + b_2 x_2(k) + b_3 x_1(k)$$

$$y(k) = (b_3 - b_0 a_3) x_1(k) + (b_2 - b_0 a_2) x_2(k) \\ + (b_1 - b_0 a_1) x_3(k) + b_0 u(k)$$

$$y(k) = \begin{pmatrix} b_3 - b_0 a_3 & b_2 - b_0 a_2 & b_1 - b_0 a_1 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix} + b_0 u(k)$$

$D_{1 \times 1}$  (pointing to  $b_0$ )

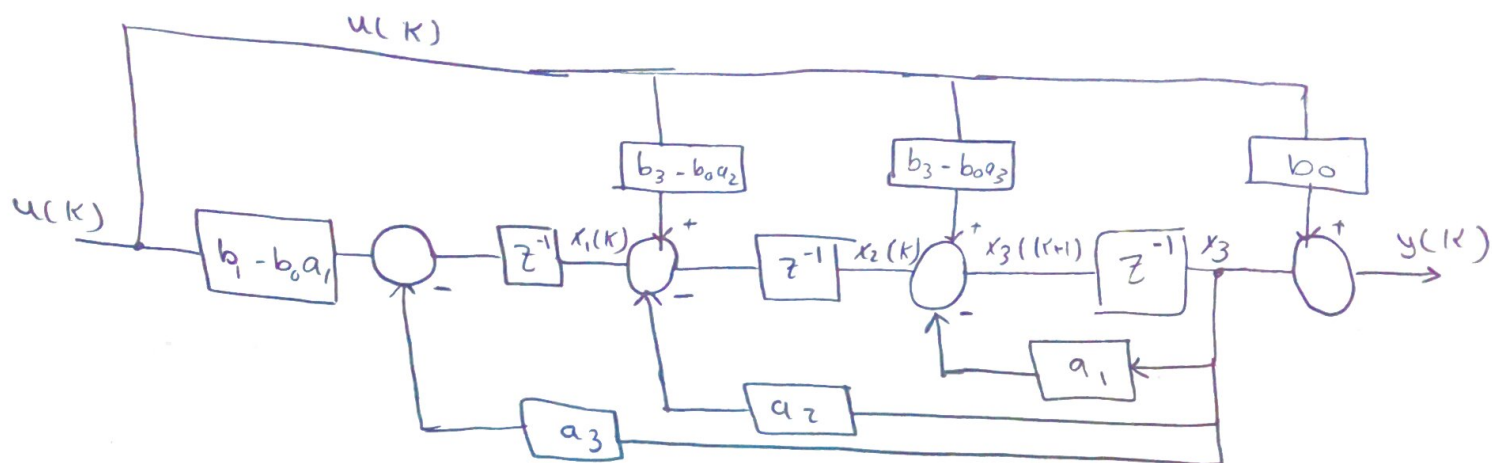


Observable form:

$$A_0 = A_c^T ; B_0 = C_c^T ; C_0 = B_c^T ; D_0 = D_c = b_0$$

$$x(k+1) = \begin{pmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{pmatrix} x(k) + \begin{pmatrix} b_3 - b_0 a_3 \\ b_2 - b_0 a_2 \\ b_1 - b_0 a_1 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x(k) + b_0 u(k)$$



Ex: T.F. = 
$$\frac{z^3 + 2z + 3}{z^3 + 4z^2 + 2z - 1}$$

Controllable Form:-

$$x(k+1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & -4 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} 4 & 0 & -4 \end{pmatrix} x(k) + u(k)$$

$\begin{matrix} \swarrow & \downarrow & \downarrow \\ 3^{-1}(-1) & 2^{-1}(2) & 0^{-1}(4) \end{matrix}$

observable form

$$x(k+1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & -4 \end{pmatrix} x(k) + \begin{pmatrix} 4 \\ 0 \\ -4 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x(k) + u(k)$$

Diagonal form

$$T.F. = \frac{2z^3 + z + 1}{z(z+0.5)(z-0.5)} \Leftarrow z(z^2 - 0.25)$$

$$\begin{array}{r} 2 \\ \hline 2z^3 - 0.25z \quad \left| \begin{array}{l} 2z^3 + z + 1 \\ 2z^3 - 0.5z \end{array} \right. \\ \hline 1.5z + 1 \end{array}$$

$$T.F. = 2 + \frac{1.5z+1}{z^3-0.25z}$$

$$= 2 + \frac{1.5z+1}{z(z+0.5)(z-0.5)} \quad \Downarrow \text{using P.F.}$$

$$D_{1 \times 1} = \textcircled{2} + \frac{A_1}{z} + \frac{A_2}{z+0.5} + \frac{A_3}{z-0.5}$$

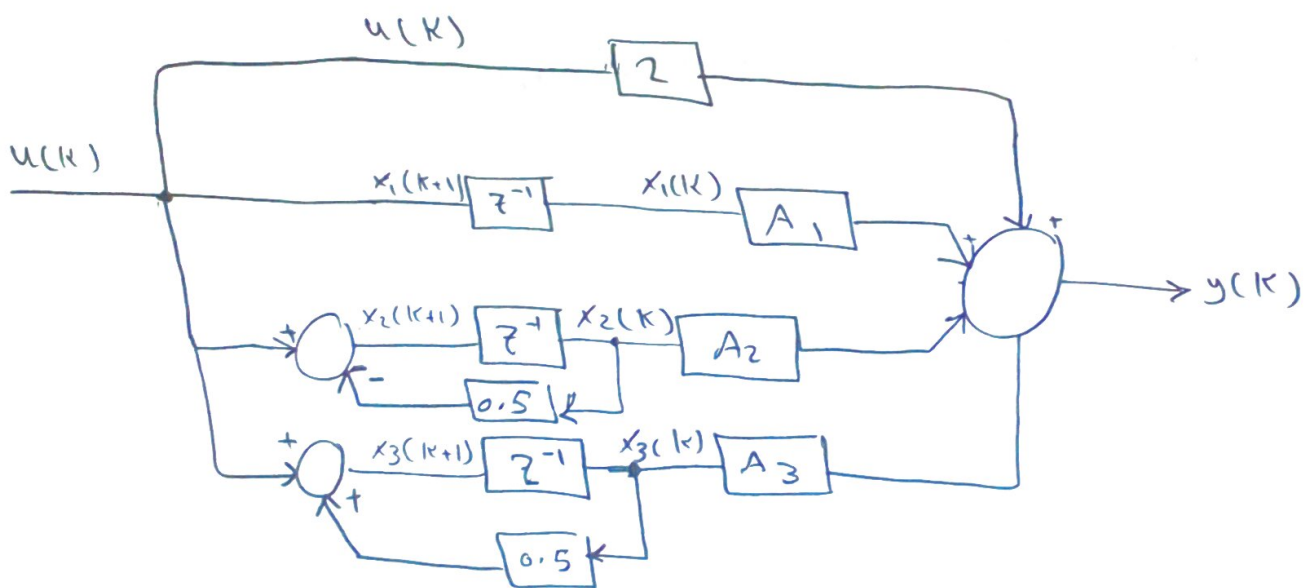
$$X(k+1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & 0.5 \end{pmatrix} X(k) + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} u(k)$$

$$A_1 = \frac{1}{(0.5)(-0.5)}$$

$$A_2 = \frac{1.5(-0.5)+1}{(-0.5)(-1)}$$

$$A_3 = \frac{(1.5)(0.5)+1}{(0.5)(1)}$$

$$y(k) = (A_1 \quad A_2 \quad A_3) x(k) + 2u(k)$$



⇒ State Space Analysis



## State Space Analysis :-

\* given 
$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$
  $A, B, C, D$  are given

① T.F.

$$x(k+1) = Ax(k) + Bu(k) \xrightarrow[(z, I, C)]{z \cdot I} zX(z) = AX(z) + Bu(z)$$

$$zX(z) - AX(z) = Bu(z)$$

$$(zI - A)X(z) = Bu(z)$$

$$X(z) = (zI - A)^{-1} Bu(z) \quad (1)$$

$$y(k) = Cx(k) + Du(k) \xrightarrow{z \cdot T}$$

$$Y(z) = CX(z) + Du(z) \quad (2)$$

$$Y(z) = C(zI - A)^{-1} Bu(z) + Du(z)$$

$$T.F. = \frac{Y(z)}{u(z)} = C(zI - A)^{-1} B + D$$

② Ch. eqn

$$|zI - A| = 0 \quad \parallel \quad T.F. = \frac{\text{num}}{\text{den}}$$

den = 0

③ check the system controllability & observability.

Controllability :-

The system is controllable if for any change of an external input, produce change in internal states of the system.

$\Rightarrow$  observability

observability :-

the system is observable if we can determine or estimate the states values from the relation between i/p and o/p or by the history information for the o/p and i/p

# Controllability Matrix ( $M_c$ )

$$M_c = (B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B) \quad n \rightarrow \text{system order}$$

for 2nd order

$$M_c = (B \quad AB)$$

for 3rd order

$$M_c = (B \quad AB \quad A^2B)$$

if  $|M_c| \neq 0$ , the system is controllable

# Observability Matrix ( $M_o$ )

$$M_o = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

for 2nd order

$$M_o = \begin{pmatrix} C \\ CA \end{pmatrix}$$

for 3rd order

$$M_o = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix}$$

$\Rightarrow$  [4] System Response

#### [4] The System Response:

if  $x(0) = 0$  (Z.I.C)

$$\textcircled{1} \text{ T.F.} = \frac{Y(z)}{u(z)} = C(zI - A)^{-1}B + D$$

$$Y(z) = [\text{T.F.}] \cdot u(z) \leftarrow \text{given}$$

$$\downarrow$$
$$z^{-1} \cdot T \Rightarrow y(k) = \checkmark$$

if  $x(0) \neq 0$  (there is a value for initial states)

$$X(k+1) = AX(k) + Bu(k)$$

$\downarrow$  Z.T

$$zX(z) - zX(0) = AX(z) + Bu(z)$$

$$zX(z) - AX(z) = zX(0) + Bu(z)$$

$$(zI - A)X(z) = zX(0) + Bu(z)$$

$$\underline{X(z) = (zI - A)^{-1}zX(0) + (zI - A)^{-1}Bu(z)}$$

$$\text{Let } \phi(z) = (zI - A)^{-1}$$

$$\underline{X(z) = \phi(z)zX(0) + \phi(z)Bu(z)}$$

$\downarrow z^{-1} \cdot T$

$$\textcircled{X(k)}$$

System Response:  $y(k) = C X(k) + D u(k)$

EX:

$$x(k+1) = \begin{pmatrix} 0 & 0.6321 \\ -1 & 1 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = (1 \quad -0.6321) x(k)$$

① Find T.F.

② Check Stability

$$T.F. = C (zI - A)^{-1} B + D = 0$$

$$(zI - A) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} 0 & 0.6321 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} z & -0.6321 \\ 1 & z-1 \end{pmatrix} \leftarrow \text{get det.}$$

$$(zI - A)^{-1} = \frac{1}{z(z-1) + 0.6321} \begin{pmatrix} z-1 & 0.6321 \\ -1 & z \end{pmatrix}$$

$$T.F. = C (zI - A)^{-1} B$$

$$= (1 \quad -0.6321) \frac{1}{z^2 - z + 0.6321} \begin{pmatrix} z-1 & 0.6321 \\ -1 & z \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{z^2 - z + 0.6321} (1 - 0.6321) \begin{pmatrix} z-1 + 0.6321 \\ -1 + z \end{pmatrix}$$

$$T.F. = \frac{z-1 + 0.6321 + 0.6321 - 0.6321z}{z^2 - z + 0.6321}$$

$$T.F. = \frac{0.3679z + 0.2642}{z^2 - z + 0.6321}$$



ch. eqn  $z^2 - z + 0.6321 = 0$

poles,  $z_{1,2} = 0.5 \pm j 0.618$

$|z_{1,2}| = 0.795 < 1 \Rightarrow$  system stable

Ex:

$$x(k+1) = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.2 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} 1 & 1 \end{pmatrix} x(k)$$

if  $x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ , find unit step response.

$$x(z) = \phi(z) z x(0) + \phi(z) B u(z)$$

$$\phi(z) = (zI - A)^{-1}$$

$$(zI - A) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} 0.8 & 0 \\ 0 & 0.2 \end{pmatrix} \\ = \begin{pmatrix} z-0.8 & 0 \\ 0 & z-0.2 \end{pmatrix}$$

$$\phi(z) = (zI - A)^{-1} = \frac{1}{(z-0.8)(z-0.2)} \begin{pmatrix} z-0.2 & 0 \\ 0 & z-0.8 \end{pmatrix}$$

$$x(z) = \begin{pmatrix} \frac{1}{z-0.8} & 0 \\ 0 & \frac{1}{z-0.2} \end{pmatrix} z \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{z-0.8} & 0 \\ 0 & \frac{1}{z-0.2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(z)$$

$u(t) = 1 \Rightarrow u(k) = 1 \Rightarrow u(z) = \frac{z}{z-1}$

$$x(z) = \begin{pmatrix} \frac{z}{z-0.8} \\ \frac{z}{z-0.2} \end{pmatrix} + \begin{pmatrix} \frac{1}{z-0.8} \\ \frac{1}{z-0.2} \end{pmatrix} \begin{pmatrix} z \\ z-1 \end{pmatrix}$$

$$X(z) = \begin{pmatrix} \frac{z}{z-0.8} \\ \frac{z}{z-0.2} \end{pmatrix} + \begin{pmatrix} \frac{1}{(z-0.8)(z-1)} \\ \frac{1}{(z-0.2)(z-1)} \end{pmatrix} z$$

$$= \begin{pmatrix} \frac{z}{z-0.8} \\ \frac{z}{z-0.2} \end{pmatrix} + \begin{pmatrix} \frac{-5}{z-0.8} + \frac{5}{z-1} \\ \frac{-1.25}{z-0.2} + \frac{1.25}{z-1} \end{pmatrix} z$$

$$z^{-1}.T \Rightarrow X(k) = \begin{pmatrix} (0.8)^k \\ (0.2)^k \end{pmatrix} + \begin{pmatrix} -5(0.8)^k + 5 \\ -1.25(0.2)^k + 1.25 \end{pmatrix}$$

$$X(k) = \begin{pmatrix} (0.8)^k - 5(0.8)^k + 5 \\ (0.2)^k - 1.25(0.2)^k + 1.25 \end{pmatrix}$$

$$= \begin{pmatrix} -4(0.8)^k + 5 \\ -0.25(0.2)^k + 1.25 \end{pmatrix}$$

The unit step response  $y(k)$  is

$$y(k) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} -4(0.8)^k + 5 \\ -0.25(0.2)^k + 1.25 \end{pmatrix}$$

$$= -4(0.8)^k - 0.25(0.2)^k + 6.25$$

$\Rightarrow$  Another example

Ex:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, C = (1 \ 0 \ 2)$$

check system's controllability and observability

$$M_c = (B \ AB \ A^2B)$$

$$AB = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ -5 \end{pmatrix}$$

$$A^2B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 11 \\ -5 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ -20 \end{pmatrix}$$

A   \*   AB

$$M_c = \begin{pmatrix} +0 & 3 & -2 \\ -1 & 11 & 7 \\ +3 & -5 & -20 \end{pmatrix} \Rightarrow |M_c| = -1 \begin{vmatrix} 3 & -2 \\ -5 & -20 \\ +3 & 3 & -2 \\ 11 & 7 \end{vmatrix}$$

$$|M_c| = 70 + 129 = 199 \neq 0$$

$\Rightarrow$  System is Controllable

$$M_o = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix}; \quad CA = (1 \ 0 \ 2) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix}$$

$$CA^2 = CA \cdot A = \begin{pmatrix} -1 & -4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -6 & -12 \end{pmatrix}$$

$\Rightarrow$  get  $|M_o|$

$$M_o = \begin{pmatrix} + & 1 & | & 0 & 2 \\ - & -1 & | & -4 & -1 \\ + & 0 & | & -6 & -12 \end{pmatrix}$$

$$= \begin{vmatrix} -4 & -1 \\ -6 & -12 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ -6 & -12 \end{vmatrix} = 54 \neq 0$$

System is observable